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On Efficiency of Some D-optimal Designs in Subset Estimation

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SUMMARY

D-optimal designs within the class of rotatable designs are considered for first-, second-, third- and fourth-order models over spherical regions. Within the class, D-efficiencies of the optimal d-th order design as a design for l-th order model for $l < d$ and $d = 2, 3$ and 4 are derived. Situations where the additional parameters due to a m-th order model ($l < m \leq d$) are also estimated and investigated too.

Keywords : D-optimal design, Efficiency, Rotatable design, Spherical region.

Introduction

Consider the problem of selecting a suitable design when the true model may be a submodel of the assumed model, i.e. the true expectation function may be only a part of the assumed expectation function. A similar problem is that of selecting an appropriate design for a given model when some parameters of the model are of little or no interest, i.e., are nuisance parameters. The D_s-optimality criterion introduced by Karlin and Studden [6] and subsequently considered by many investigators deals with D-optimality for $s(<p)$ of the parameters where p is the total number of parameters in the assumed model. The D_s- efficiency of a design is its efficiency under D_s-optimality criterion. Clearly appropriate versions of these functions are suitable design criterion when the true model is a submodel of the assumed model. Note that the available literature seems to have very few results on D_s-optimal designs for polynomial models in more than one variable.

In what follows we consider multivariate polynomial regression in spherical regions and investigate the consequence of using a D-optimal d-th order design when the parameters of interest are only those belonging to a model of order l (<d) for d=2,3, and 4. We study the efficiency of the design with respect to the D-optimal design for the parameters of interest. Having performed a design for a d-th order model when his interest is in a l-th order model, the experimenter has several options. One option is to estimate the parameters of the l-th order model only, ignoring the other parameters completely. More generally he may fit a $(l+i)$-th order model ($i = 0, 1, 2, ..., d-l$) but make inferences only about

the parameters of the l-th order model. This the experimenter might do if he learns about the true model only after estimating all the parameters in the l-th order model. For each i, the consequences would be quite different and distinction must be made between these situations. The cases, $i = o$ and $i = d - l$ are important from a practical point of view. The other cases, although possibly not so relevant, have been included for the sake of completeness of the study from a theoretical stand point.

2. *Preliminaries*

Let there be K quantitative factors $x_1,...,x_k$. Suppose the response Y(x) at point $x = (x_1,...,x_k)'$ in the factor space is assumed to have an expectation function given by

$$E\{Y(X)\} = \beta_o + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \sum_{j=1}^{i} \beta_{ij} x_i x_j + ... \qquad (1)$$

where the right hand side of (1) contains all the $\binom{k+d}{d}$ terms of a polynomial of order d. The observations on the response are assumed to be uncorrelated and homoscedastic, the common variance being, without loss of generality, taken to be unity.

A design ζ of order d is a probability measure on the experimental region z which allows estimation of all the parameters in a d-th order model. Writing $E\{y(x)\} = f'(x)\beta$ the information matrix of ζ is given by $M(\zeta) = \int_Z f(x) f'(x) \zeta(dx)$. If N trials are performed in accordance with ζ then the dispersion matrix of the least squares estimator $\hat{\beta}$ of β given by $N^{-1} M^{-1}(\zeta)$. Under D- optimality criterion the objective is to minimize the generalized variance of $\hat{\beta}$ i.e. to maximize $|M(\zeta)|$. The D-efficiency of a design ζ is $\{|M(\zeta)|/|M(\zeta^*)|\}^{1/p}$ where ζ^* is the D-optimal design and p is the number of parameters in the model.

Now suppose β' is partitioned as $[\beta'_{(1)}, \beta'_{(2)}]$ where $\beta_{(1)}$ is the vector of parameters of interest, with corresponding partitioning of $f'(x)$ as $[f'_{(1)}(x), f'_{(2)}(x)]$. Then

$$M(\zeta) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where $M_{ij} = \int_z f_{(i)}(x) f'_{(j)}(x) \zeta(dx)$ $(i, j = 1, 2)$ and $M_{12} = M'_{21}$.

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CHANGE IN DATES OF CONFERENCE

Due to unavoidable circumstances, the 49th Annual Conference of the Society will be held from 20 to 22 December, 1995 instead of from 19 to 21 December, 1995. The programme of the Conference will, however, remain the same but for shifting by a day. The inconvenience caused, if any, is regretted.

Kindly bring this to the notice of your colleagues.

P.R. SREENATH
Secretary

If $\beta_{(1)}$ alone is estimated the corresponding information matrix is M_{11} while if both $\beta_{(1)}$ and $\beta_{(2)}$ are estimated the information matrix for $\beta_{(1)}$ is $(M_{11} - M_{12}\, M_{22}^{-1}\, M_{21})$.

Suppose that the x_i's are suitably standardized so that z is a hypersphere which without loss of generality is taken to be centered at the origin and of unit radius, i.e. $z = \{x : x'x \le 1\} = B_k^{(1)}$. Let $S_k(t^{1/2})$ denote the spherical shell of radius $t^{1/2}$ centred at the origin, i.e. $S_k(t^{1/2}) = \{x : x'x = t\}$

In what follows we shall restrict attention to the class of rotatable designs which were introduced by Box and Hunter [1] and are known to be only admissible ones for spherical regions under D- and many commonly used criteria (see Kiefer, [7]). For a rotatable design of order d

$$\int_z x_1^{\alpha_1} \ldots x_k^{\alpha_k}\, \zeta(dx) = \begin{cases} 0 & \text{if one or more of the } \alpha_i \text{ are odd,} \\ \lambda_\alpha \prod_{i=1}^{k} \dfrac{\alpha_i !}{2^{\alpha/2} \prod_{i=1}^{k} (\alpha_i/2)!} & \text{if all of the } \alpha_i \text{ are even} \end{cases}$$

where $\alpha = \sum_1^k \alpha_i \le 2d$ and $\lambda_2, \lambda_4, \ldots, \lambda_{2d}$ are moments of the design ζ. Thus, for a first-order rotatable design, $M(\zeta) = \text{Diag}\{1, \lambda_2 I_k\}$ and for a second-order rotatable design, writing $f'(x) = (1, x_1^2, \ldots, x_k^2; x_1, \ldots, x_k; x_1 x_2, \ldots, x_{k-1} x_k)$ we obtain $M(\zeta) = \text{Diag}\{A_1, A_2, A_3\}$ where

$$A_1 = \begin{bmatrix} 1 & \lambda_2 E_{1,k} \\ \lambda_2 E_{k,1} & \lambda_4 (2I_k + E_{k,k}) \end{bmatrix}, A_2 = \lambda_2 I_k, A_3 = \lambda_4\, I_{\binom{k}{2}}.$$

In the above and what follows I_a denotes the a x a identity matrix and $E_{a,b}$ denotes the a x b matrix of 1's.

For a third-order rotatable design, $M(\zeta) = \text{Diag}\{A_1, A_4, A_3, A_5\}$ where $A_5 = \lambda_6 I_{\binom{k}{3}}$, $A_4 = \text{Diag}\{F_1, \ldots F_k\}$ and $F_i = F^*$ (i=1,...,k), F^* being given by

$$F^* = \begin{bmatrix} \lambda_2 & 3\lambda_4 & \lambda_4 E_{1,k-1} \\ 3\lambda_4 & 15\lambda_6 & 3\lambda_6 E_{1,k-1} \\ \lambda_4 E_{k-1,1} & 3\lambda_6 E_{k-1,1} & \lambda_6 (2I_{k-1} + E_{k-1,k-1}) \end{bmatrix}.$$

This blocking of M (ζ) arises from partitioning f′ (x) into four component sets of terms: (1) $\{1, x_1^2, ..., x_k^2\}$, (2) $\{x_i,\ x_i x_1^2,..., x_i x_k^2$; there are k such sets$\}$ (3) $\{x_i x_j\ (i<j)\}$, (4) $\{x_i\ x_j\ x_l\ (i, j, l)\}$

Similarly, by arranging the terms in f(x) suitably, M (ζ) for a fourth- order rotatable design may be written as a partitioned matrix with five non-zero blocks along the principal diagonal and zeros elsewhere (Draper and Herzberg [2]). The non-zero blocks arise from intersection of rows and columns corresponding to the five sets of terms :

(1) $\{1, x_1^2,.., x_k^2, x_1^4,..., x_k^4,\ x_1^2\, x_2^2,..., x_{k-1}^2\, x_k^2\}$,

(2) $\{x_i, x_i\, x_1^2,..., x_i\, x_k^2$; there are k such sets$\}$,

(3) $\{x_i\, x_j, x_i\, x_j\, x_1^2,.., x_i\, x_j\, x_k^2\ (i < j)$; there are $k\ (k-1)/2$ such sets$\}$,

(4) $\{x_i\, x_j\, x_l\ \ (i < j < l)\}$ (for $k \geq 3$),

(5) $\{x_i\, x_j\, x_u\, x_v\ \ (i < j < u < v)\}$ (for $k \geq 4$).

Thus for a fourth-order rotatable design ζ, M (ζ) = Diag $\{M_1, M_2, M_3, M_4, M_5\}$ where $M_4 = \lambda_6\, I_{\binom{k}{3}}$, $M_5 = \lambda_8\, I_{\binom{k}{4}}$, $M_2 = I_k \otimes M_2^*$, $M_3 = I_{\binom{k}{2}} \otimes M_3^*$.

Here ⊗ denotes the Kronecker product, $M_2^* = F^*$ and

$$M_3^* = \begin{bmatrix} \lambda_4 & 3\,\lambda_6 & 3\,\lambda_6 & \lambda_6\, E_{1,k-2} \\ 3\,\lambda_6 & 15\,\lambda_8 & 9\,\lambda_8 & 3\,\lambda 8\, E_{1,k-2} \\ 3\,\lambda_6 & 9\,\lambda_8 & 15\lambda_8 & 3\,\lambda_8\, E_{1,k-2} \\ \lambda_6\, E_{k-2,1} & 3\,\lambda_8\, E_{k-2,1} & 3\,\lambda_8\, E_{k-2,1} & \lambda_8\, (2I_{k-2} + E_{k-2,k-2}) \end{bmatrix}.$$

The matrix M_1 has a more complex structure and may be expressed as

$$M_1 = \begin{bmatrix} 1 & \lambda_2\ E_{1,k} & 3\,\lambda_4\, E_{1,k} & \lambda_4\, E_1, k' \\ \lambda_2\, E_{k,1} & \lambda_4\, (2I_k + E_{k,k}) & \lambda_6\, (12I_k + 3E_{k,k}) & \lambda_6\, (2B + E_{k,k'}) \\ 3\,\lambda_4\, E_{k,1} & \lambda_6\, (12I_K + 3E_{k,k} & \lambda_8\, (96I_k + 9E_{k,k}) & \lambda_8\, (12B + 3E_{k,k'}) \\ \lambda_4\, E_{k',1} & \lambda_6\, (2B + E_{k',k}) & \lambda_8\, (12B + 3E_{k',k}) & \lambda_8\, (9I_k' + 3B_1 + B_2) \end{bmatrix}.$$

where $k' = k\ (k-1)/2$, $B = ((b_{l,\,ij}))$, $b_{l,ij} = 1$ if $l \in \{i, j\}$ and $b_{l,ij} = 0$ otherwise ($l = 1, ...,k$; $1 \leq i \leq j \leq k$). Further B_1, B_2 are association matrices corresponding to first and second associate relationships respectively of a triangular type partially balanced association scheme in k′ symbols (see, Huda and Mukerjee [4]).

3. *D-optimal designs*

In what follows wee shall consider the D-optimal designs within the class of rotatable designs.

3.1. First-order designs

For a first-order rotatable design ζ, $|M(\zeta)| = \lambda_2^k = V_1(\zeta)$ which is maximized by ζ_1 having $\lambda_2 = 1/k$ so that the D-efficiency of ζ is $\{V_1(\zeta)/V_1(\zeta_1)\}^{1/(k+1)} = (k\lambda_2)^{k/(k+1)}$..

3.2. Second-order designs

For a second-order rotatable design ζ, $|M((\zeta)| = 2^{k-1} \lambda_2^k \lambda_4^{(k-1)(k+2)/2} \{(k+2)\lambda_4 - k\lambda_2^2\} = V_2(\zeta)$, which is readily seen to be maximized by ζ_2 having $\lambda_4 = \lambda_{4(2)}$, $\lambda_2 = \lambda_{2(2)}$ where $\lambda_{4(2)} = \lambda_{2(2)}/(k+2)$ and $\lambda_{2(2)} = (k+3)/\{(k+1)(k+2)\}$. Therefore, D-efficiency of ζ is

$$V_2(\zeta)/V_2(\zeta_2)\}^{2/\{(k+1)(k+2)\}} = [\{(k+1)(k+2)\lambda_2/(k+3)\}^k$$
$$\{(k+1)(k+2)^2\lambda_4/(k+3)\}^{(k-1)(k+2)/2}$$
$$\{(K+2)\lambda_4 - k\lambda_2^2\}(k+1)^2(k+2)^2/\{2(k+3)\}]^{2/(k+1)(k+2)}.$$

3.3. Third-order designs

For a third-order rotatable design ζ,

$$|M(\zeta)| = 2^{(k+1)(k-1)}\, 3^k\, \lambda_4^{(k-1)(k+2)/2}\, \{k+2)\lambda_4 - k\lambda_2^2\}$$
$$\lambda_6^{k(k-1)(k+4)/6}\{(k+4)\lambda_2\lambda_6 - (k+2)\lambda_4^2\}^k = V_3(\zeta).$$

It can be seen that for given $\lambda_2$and λ_4, is maximized by taking $\lambda_6$as large as possible. It follows that the D-optimal design ζ_3 will be of the type putting a mass, say (1–w) and a mass w uniformly over $S_k(1)$ and $S_k(\rho^{1/2})$, respectively, for some $\rho < 1$. For this type of design, $\lambda_{2i} = (1 - w + w\rho^i)/\{k(k+2)...(k+2i-2)\}$ i=1, 2, 3 so that $\{(k+2)\ \lambda_4 - k\lambda_2^2\} = w(1-w)(1-\rho)^2/k$ and $\{(k+4)\lambda_2\lambda_6 - (k+2)\lambda_4^2\} = w(1-w)\rho(1-\rho)^2/\{k^2(k+2)\}$. Then $|M(\zeta)|$ has to be maximized as a function of ρ and w over $o < w, \rho < 1$. However, the maximizing values of ρ and w can not be obtained algebraically. The values were obtained numerically in Galil and Kiefer [3} and are reproduced below. The D-efficiency of a third-order rotatable design is therefore

$$\left(\frac{V_3(\zeta)}{V_3(\zeta_3)}\right)^{6/\{(k+3)(k+2)(k+1)\}}$$

$$= \left[\left(\frac{K(k+2)\lambda_4}{(1-w+w\rho^2)}\right)^{(k-1)(k+2)/2}\right.$$

$$k\left(\frac{(k+2)\lambda_4^2-k\lambda_2^2}{w(1-w)(1-\rho)^2}\right)\left(\frac{k(k+2)(k+4)\lambda_6}{(1-w+w\rho^3)}\right)^{k(k-1)(k+4)/6}$$

$$\left.\left[\frac{\{(k+4)\lambda_2\lambda_6-(k+2)\lambda_4^2\}k^2(k+2)}{w(1-w)\rho(1-\rho)^2}\right]^k\right]^{6/\{(k+3)(k+2)(k+1)\}}$$

where w and ρ correspond to the D-optimal design ζ_3 and are as in Table 1.

Tablel. D-optimal values of w, $\rho^{1/2}$

K	2	3	4	5	6	7	8	9	10
w	0.308	0.208	0.150	0.113	0.088	0.070	0.058	0.048	0.041
$\rho^{1/2}$	0.515	0.544	0.560	0.569	0.576	0.580	0.584	0.586	0.588

3.4. Fourth-order designs.

For a fourth-order rotatable design ζ the expression for $|M(\zeta)|$ is rather complicated and were derived by Patel and Arapkoske [9] and also by Huda and Mukerjee [4]. Huda and Mukerjee [5] subsequently showed that the D-optimal design ζ_4 puts masses w_0, w_1 and w_2 uniformly spread over $S_k(0)$, $S_k(t^{1/2})$ and $S_k(1)$, respectively where $w_0 = 1-(w_1+w_2)$ and w_1, w_2 and $t^{1/2}$, w_1, w_2 are as in Table 2 which follows. Note that for designs of this type $\lambda_{2i} = (w_2+w_1t^i) / \{k(k+2)\ldots(k+2i-2)\}$ i = 1, 2, 3, 4 and hence

$$(k+4)\lambda_2\lambda_6-(k+2)\lambda_4^2 = w_1w_2t(1-t)^2/\{k^2(k+2)\},$$

$$(k+6)\lambda_4\lambda_8-(k+4)\lambda_6^2 = w_1w_2t^2(1-t)^2/\{k^2(k+2)^2(k+4)\},$$

$$(k+6)\lambda_2\lambda_8-(k+2)\lambda_4\lambda_6 = w_1w_2t(1-t)^2(1+t)/\{k^2(k+2)(k+4)\}.$$

Table 2. D-optimal values of t, w_1 and w_2

K	2	3	4	5	6	7	8	9	10
t	.4602	.4794	.4920	.5007	.5071	.5120	.5158	.5188	.5212
w_1	.3439	.2704	.2129	.1704	.1388	.1150	.0966	.0823	.0709
w_2	.5894	.7010	.7728	.8217	.8564	.8820	.9013	.9163	.9281

4. *D-efficiencies*

In this section we study the efficiency of the D-optimal designs in estimating only a subset of the parameters from the assumed model.

Consider the following scenario. Assume that $a \geq b \geq c$ with at least one strict inequality and suppose that

(i) the design ζ_a which is D-optimal for the a-th order model is employed, and

(ii) the b-th order model is fitted, and

(iii) inferences are made only about the parameters of the c-th order model.

Let E(a, b, c,) be the D-efficiency of ζ_a in this role. Then it immediately follows that

$$E(2,1,1,) = \{V_1(\zeta_2)/V_1(\zeta_1)\}^{1/(k+1)}$$

$$= [\{k(k+3)\}/\{(k+1)(k+2)\}]^{k/(k+1)},$$

$$E(3,1,1,) = \{V_1(\zeta_3)/V_1(\zeta_1)\}^{1/(k+1)} = (1-w+w\rho)^{k/(k+1)},$$

$$E(4,1,1,) = \{(V_1(\zeta_4)/V_1(\zeta_1)\}^{1/(k+1)} = (w_2+w_1 t)^{k/(k+1)}.$$

When a rotatable design of order $a \geq 2$ is used to estimate only the parameters of a second-order model, the relevant information matrix for the parameters of a first-order model is

$$\text{Diag}\ \{\{(k+2)\lambda_4 - k\lambda_2^2\}/\{(k+2)\lambda_4\},\ \lambda_2 I_k\}.$$

It follows that

$$E(2,2,1,) = [2/\{(k+1)(k+2)\}]^{1/(k+1)} E(2,1,1),$$

$$E(3,2,1,) = \{w(1-w)(1-\rho)^2/(1-w+w\rho^2)\}^{1/(k+1)} E(3,1,1),$$

$$E(4,2,1,) = \{1-(w_2+w_1 t)^2/(w_2+w_1 t^2)\}^{1/(k+1)} E(4,1,1).$$

Also, it is readily seen that

$$E(3, 2, 2) = \{V_2(\zeta_3)/V_2(\zeta_2)\}^{2/(k+1)(k+2)\}}$$

$$= [(1 - w + w\rho)^k (1 - w + w\rho^2)^{(k-1)(k+2)/2} w(1 - w)(1 - \rho)^2$$

$$\left(\frac{(k+1)(k+2)}{k(k+3)}\right)^{k(k+3)/2} (k+1)(k+2)/2]^{2/\{(k+1)(k+2)\}}$$

$$E(4, 2, 2) = \{V_2(\zeta_4)/V_2(\zeta_2)\}^{2/(k+1)(k+2)\}}$$

$$= [(w_2 + w_1 t)^k (w_2 + w_1 t^2)^{(k-1)(k+2)/2} (w_2 + w_1 t^2)(w_2 + w_1 t)^2\}$$

$$\left(\frac{(k+1)(k+2)}{k(k+3)}\right)^{k(k+3)/2} (k+1)(k+2)/2]^{2/\{(k+1)(k+2)\}}$$

When a rotatable design of order $a \geq 3$ is used to estimate only the parameters of a third-order model, the relevant information matrix for the parameters of a second-order model is the block diagonal matrix given by

$$\text{Diag}\{A_1, A_3, [\{(k+4)\lambda_2\lambda_6 - (k+2)\lambda_4^2\}/\{(k+4)\lambda_6\}] I_k\}$$

where A_1 and A_3 are as defined in Section 2. It follows that

$$E(3, 3, 2,) = [\{w(1 - w)\rho(1 - \rho)^2/(1 - w + w\rho^3)\}^k (1 - w + w\rho^2)^{(k-1)(k+2)/2}$$

$$w(1 - w)(1 - \rho)^2 \{(k+1)(k+2)/k(k+3)\}^{k(k+3)/2}$$

$$(k+1)(k+2)/2]^{2/(k+1)(k+2)}$$

$$= [w(1 - w)\rho(1 - \rho)^2/\{(1 - w + w\rho)(1 - w + w\rho^3)\}]^{2k/(k+1)(k+2)} E(3, 2, 2),$$

$$E(4, 3, 2) = [\{w_1 w_2 t(1 - t)^2/(w_2 + w_1 t^3)\}^k (w_2 + w_1 t^2)^{(k-1)(k+2)/2}$$

$$\{(w_2 + w_1 t^2) - (w_2 + w_1 t)^2\} (k+1)(k+2)/k(k = 3)\}^{k(k+3)/2}$$

$$(k+1)(k+2)/2]^{2/(k+1)(k+2)}$$

$$= [w_1 w_2 t(1 - t)^2/\{(w_2 + w_1 t)(w_2 + w_1 t^3)\}]^{2k/(k+1)(k+2)} E(4, 2, 2,).$$

Similarly it can be seen that

$$E(4, 3, 3) = \{V_3(\zeta_4)/V_3(\zeta_3)\}^{6/\{(k+3)(k+2)(k+1)\}}$$

$$= [\{(w_2 + w_1 t^2)/(1 - w + w\rho^2)\}^{(k-1)(k+2)/2}$$

$$\{\{(w_2 + w_1 t^2) - (w_2 + w_1 t)^2\}/w(1 - w)(1 - \rho)^2\}$$

$$\{(w_2 + w_1 t^3)/(1 - w + w\rho^3)^{k(k-1)(k+4)/6}$$

$$\{w_1 w_2 t (1-t)^2 / w (1-w) \rho (1-\rho)^2\}^k]^{6/\{(k+3)(k+2)(k+1)\}}.$$

When a rotatable design of order $a \geq 3$ is used to estimate only the parameters of a third-order model the information matrix for the parameters of a first-order model is

$$\text{Diag}\{\{(k+2)\lambda_4 - k\lambda_2^2\}/\{k+2)\lambda_4\}, [\{(k+4)\lambda_2 \lambda_6 - (k+2)\lambda_4^2\}/\{(k+4)\lambda_6\}]I_k\}.$$

It follows that

$$E(3,3,1,) = [(w(1-w)(1-\rho^2)/(1-w+w\rho^2))\cdot$$

$$\{w(1-w)\rho(1-\rho)^2/(1-w+w\rho^3\}^k]^{1/(k+1)}$$

$$= [w(1-w)\rho(1-\rho)^2/\{(1-w+w\rho^3)/(1-w+w\rho)\}]^{k/(k+1)} E(3,2,1).$$

$$E(4,3,1,) = [\{1-(w_2+w_1 t)^2/(w_2+w_1 t^2)\}$$

$$\{w_1 w_2 t(1-t)^2/(w_2+w_1 t^3)^k]^{1/(k+1)}$$

$$= [w_1 w_2 t (1-t)^2/\{(w_2+w_1 t)(w_2+w_1 t^3)\}]^{k/(k+1)} E(4,2,1).$$

It remains to derive E (4, 4, c,) for c = 1, 2, 3,. In order to do so we need to obtain inverses of some of the blocks along the diagonal of the information matrix of a fourth-order rotatable design. In general these, particularly the inverses of the blocks from M_1 defined earlier, are quite complicated. The following approach, also used in Mukerjee and Huda [8] allows very simple expressions for the required inverses. Define

$$A = -(k+6)\lambda_8^{-1}\begin{bmatrix}\lambda_4 (k+4)^{-1} 1_k, & \lambda_6 I \\ 2\lambda_4 (k+4)^{-1} 1_{k'}', & \lambda_6 B'\end{bmatrix}, C = \begin{bmatrix} I_k & 0 \\ B' & I_{k'}\end{bmatrix},$$

$$W = \begin{bmatrix} I_{k+1} & 0 \\ A & C\end{bmatrix}$$

where $k' = k(k-1)/2$ and $1_a = E_a, 1$.

Then it can be seen that $W M_1 W$ = Diag $\{Z, T\}$ so that

$$M_1^{-1} = \begin{bmatrix} z^{-1} & z^{-1}A' \\ A z^{-1} & CT^{-1}C' + A z^{-1} A'\end{bmatrix}$$

where

$$Z = \begin{bmatrix} z_{11} & z_{12}\, 1_k' \\ z_{12}\, 1_k & z_{22}\,(2I_k + J) \end{bmatrix}, T = \lambda_8 \begin{bmatrix} (k+4))\,(k+6)\,(2I_k + J), & (k+6)\,(2B+E) \\ (k+6)\,(2B+E), & 9I_{k'} + 3B_1 + B_2 \end{bmatrix}$$

with

$$z_{11} = 1 - k\,(k+2)\,\lambda_4^2/\{\,(k+4)\,(k+6)\,\lambda_8\,\},\ z_{22} = \lambda_4 - (k+4)\,\lambda_6^2/\{(k+6)\,\lambda_8\}$$

$$z_{12} = \lambda_2 - (k+2)\,\lambda_4\,\lambda_6/\{\,(k+6)\,\lambda_8\,\},\ \text{and } J = E_{k,k},\ E = E_{k,k'}.$$

Let z^{ij} denote the (i, j)–th element of z^{-1}. Then $z^{11} = (k+2)\,z_{22}/\{(k+2)\,z_{11}\,z_{22} - k\,z_{12}^2\}$. It follows that when a fourth-order rotatable design is used to estimate all the parameters of fourth-order model the relevant information matrix for the parameters of a first-order model is

$$\text{Diag}\ \{(z^{11})^{-1}, [\{\,(k+4)\,\lambda_2\,\lambda_6 - (k+2)\,\lambda_4^2\}/(k+4)\,\lambda_6\,\}]\ I_k\}.$$

Now, for the D-optimal design ζ_4, it can be seen after some algebra that $z^{11} = w_0^{-1}$. Hence

$$E\,(4, 4, 1) = [w_0\,\{w_1\,w_2 t\,(1-t)^2/(w_2 + w_1 t^3)\,\}^k\,]^{1/(k+1)}$$

$$= [(1 - w_1 - w_2)/\{1 - (w_2 + w_1 t^2)/(w_2 + w_1 t^2)\}]^{1/(k+1)}\,E\,(4, 3, 1)$$

Similarly, when a fourth-order rotatable design is used to estimate all the parameters of a fourth-order model, the information matrix for the parameters of a second-order model is

$$\text{Diag}\ \{Z, \left[\frac{(k+4)\,\lambda_2\,\lambda_6 - (k+2)\,\lambda_4^2}{(k+4)\,\lambda_6}\right] I_k', \left[\frac{(k+6)\,\lambda_4\,\lambda_8 - (k+4)\lambda_6^2}{(k+6)\,\lambda_8}\right] I_k'\}$$

Hence

$$E\,(4, 4, 2) = \left[(1 - w_1 - w_2)\left\{\frac{w_1\,w_2\,(1-t)^2\,(k+1)\,(k+2)}{k\,(k+3)}\right\}^{k\,(k+3)/2k\,(k+2)} \left\{\frac{(k+1)\,(k+2)}{2\,(w_2 + w_1 t^3)^k\,(w_2 + w_1 t^4)}\right\}^{k\,(k+1)/2}\right]^{2/(k+1)\,(k+2)}$$

$$= \left[\frac{(1 - w_1 - w_2) \left(\dfrac{w_1 w_2 t^2 (1 - t)^2}{(w_2 + w_1 t^4)} \right)^{k(k+1)/2}}{(w_2 + w_1 t^2)^{(k-1)(k+2)/2} \{(w_2 + w_1 t^2) - (w_2 + w_1 t)^2\}} \right]^{2/(k+1)(k+2)} \qquad E(4, 3, 2)$$

When a fourth-order rotatable design is used to estimate all the parameters of a fourth-order model, the information matrix for the parameters of a third-order model is

$$\text{Diag}\ \{Z, M_2, [\{(k+6)\lambda_4 \lambda_8 - (k+4)\lambda_6^2\}/(k+6)\lambda_8\}]\ I_k', M_4\}$$

Therefore, it can be seen that

$$E(4,4,3) = \left[\left(\frac{w_0 (1 - w + w\rho^2)}{w(1-w)(1-\rho)^2} \right) \left(\frac{w_1 w_2 t^2 (1-t)^2}{(w_2 + w_1 t^4)(1 - w + w\rho^2)} \right)^{k(k+1)/2} \right.$$

$$\left. \left\{ \frac{(w_2 + w_1 t^3)}{(1 - w + w\rho^3)} \right\}^{k(k-1)(k+4)/6} \left\{ \frac{w_1 w_2 t (1-t)^2}{w(1-w)\rho(1-\rho)^2} \right\}^{k} \right]^{6/(k+3)(k+2)(k+1)}$$

$$= \left[\frac{(1 - w_1 - w_2) \left(\dfrac{w_1 w_2 t^2 (1-t)^2}{(w_2 + w_1 t^2)(w_2 + w_1 t^4)} \right)^{k(k+1)/2}}{\left(1 - \dfrac{(w_2 + w_1 t)^2}{(w_2 + w_1 t^2)} \right)} \right]^{6/(k+3)(k+2)(k+1)} \qquad E(4, 3, 3)$$

The numerical values of E(a,b,c) were computed for $k = 2$ upto $k = 10$ and the results are displayed in Table 3.

5. *Comments*

The efficiencies E(a, c, c) and E(a, a, c) are highly relevant from a practical point of view. Since statistical models are usually adhoc in nature, the experimenter may learn about the true order c of the model only after performing the experiments in accordance with the optimal design ζ_a for his model assumed to be of order a. If he does not have the resources to run further experiments he would obviously use the data already obtained and fit a model of order c. Then E(a, c, c) gives an idea of how well he will be doing in this situation.

Similarly, E(a, a, c) becomes relevant when the experimenter learns about the true model only after completing his estimation of the assumed model and

Table 3. Efficiencies (in %) of D-optimal designs

K	2	3	4	5	6	7	8	9	10
E (2, 1, 1)	88.55	92.40	94.63	96.02	96.93	97.57	98.02	98.36	98.62
E (3, 1, 1)	84.28	88.80	91.67	93.59	94.94	95.92	96.59	97.16	97.56
E (4, 1, 1)	82.38	87.01	90.08	92.19	93.69	94.81	95.64	96.30	96.82
E (2, 2, 1)	48.73	51.96	55.06	57.81	60.22	62.34	64.21	65.88	67.38
E (3, 2, 1)	45.87	50.03	53.77	56.99	59.70	62.02	64.11	65.89	67.52
E (4, 2, 1)	44.34	48.42	52.27	55.63	58.54	61.03	63.23	65.12	66.82
E (3, 2, 2)	89.78	92.48	94.30	95.56	96.46	97.14	97.59	97.98	98.26
E (4, 2, 2)	85.03	88.06	90.48	92.29	93.67	94.71	95.53	96.17	96.68
E (3, 3, 2)	34.46	33.97	35.10	36.73	38.53	40.30	42.16	43.82	45.51
E (4, 3, 2)	32.98	33.67	35.46	37.53	39.63	41.66	43.58	45.40	47.10
E (4, 3, 3)	93.08	95.30	96.59	97.40	97.93	98.31	98.59	98.80	98.97
E (3, 3, 1)	6.76	4.09	2.77	2.00	1.52	1.18	0.96	0.79	0.67
E (4, 3, 1)	6.67	4.38	3.15	2.39	1.88	1.51	1.25	1.05	0.89
E (4, 4, 1)	5.03	3.24	2.32	1.77	1.40	1.13	0.95	0.80	0.69
E (4, 4, 2)	5.06	2.99	2.02	1.48	1.13	0.90	0.73	0.61	0.51
E (4, 4, 3)	30.22	28.40	27.70	28.95	29.93	31.08	32.32	33.59	34.86

is either unwilling or unable to carry out further computations that would be necessary to revise his estimates in the light of fresh information.

Although E(a, b, c) with a>b>c is possibly less important, none the less it could be relevant in some situations. For example, when the experimenter starts with a model of order a, performs the experiments and just before beginning the estimation is told that the model should be of order b and after the estimation is told that the true order is c, E(a, b, c) gives a measure of how well he will do if he does not revise his estimates in the light of latest information.

The determinant of the relevant information matrix strictly decreases as we increase b from c to a. This happens as a penalty for estimation of increasing number of nuisance parameters. Thus, as expected, the efficiency function E(a, b, c) is a strictly decreasing function of b. Also note that E(a, b, c) < E(a′, b, c) when a > a′ for most values of k under consideration.

The results presented in this paper should be useful to the experimenter, since these inform him how well he stands to do if he does not start afresh in the light of most recent information that he has received about the model.

Figure 1 which follows provides a graphical display of all the efficiencies derived.

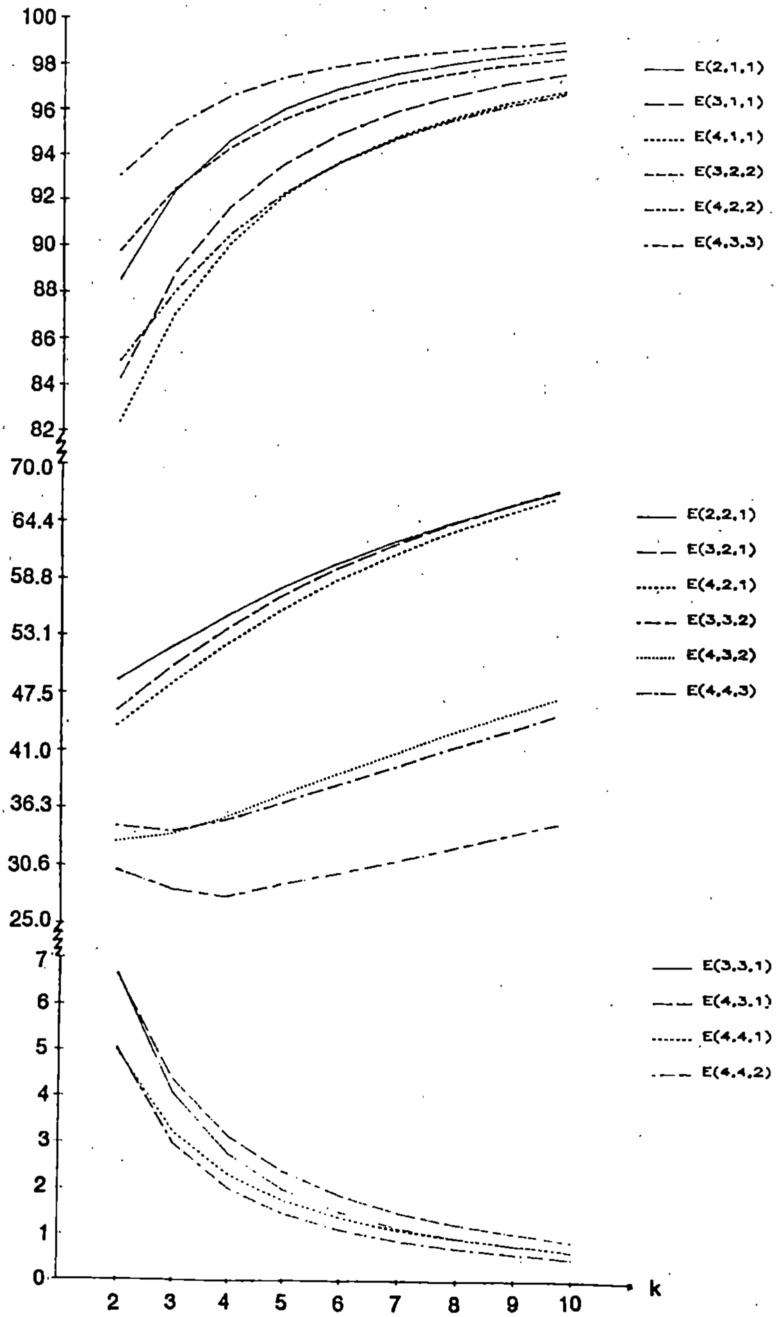

Fig. I. Graphical display of efficiency of D-optimal designs

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